The Solution of the Linear Fractional Partial Differential Equations Using the Homotopy Analysis Method

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In this paper, the homotopy analysis method is applied to solve linear fractional problems. Based on this method, a scheme is developed to obtain approximation solution of fractional wave, Burgers, Korteweg-de Vries (KdV), KdV-Burgers, and Klein-Gordon equations with initial conditions, which are introduced by replacing some integer-order time derivatives by fractional derivatives. The fractional derivatives are described in the Caputo sense. So the homotopy analysis method for partial differential equations of integer order is directly extended to derive explicit and numerical solutions of the fractional partial differential equations. The solutions are calculated in the form of convergent series with easily computable components. The results of applying this procedure to the studied cases show the high accuracy and efficiency of the new technique.

Key words: Homotopy Analysis Method (HAM); Analytical Solution; Fractional Partial Differential Equations (FPDEs).

1. Introduction

In recent years, considerable interest in fractional partial differential equations (FPDEs) has been stimulated due to their numerous applications in the areas of physics and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional partial differential equations [2 – 4]. Also, fractional partial differential equations have been found to be effective to describe some physical phenomena such as damping laws, rheology, diffusion processes, and so on. In general, there exists no method that yields an exact solution for a fractional partial differential equation. Since most of the nonlinear fractional partial differential equations cannot be solved exactly, thus approximate and numerical methods must be used. Author of [5] found an approximate solution of a nonlinear equation with Riemann-Liouville's fractional derivatives by He's variational iterational method. Several methods have been used to solve fractional partial differential equations, such as Adomian's decomposition method (ADM) [6, 7], Fourier transform method [8], Laplace transform method [2, 3, 9], and so on. Some fundamental works on various aspects of the fractional calculus are given by Abbasbandy [5], Al-Khaled and Momani [10], Caputo [11], Debnath [12], Diethelm et al. [13], Jafari and Seifi [14, 15], Hayat et al. [16], Khan and Hayat [17], Kilbas and Trujillo [18], Kiryakova [19], Oldham and Spanier [20], Ray and Bera [21], Shawagfeh [22], Song and Zhang [23, 24], Xu and Jie [25], Momani and his co-authors [26 – 28], etc. The interested reader can see [29 – 36] for more application of the method. Moreover, there are some recent attempts to applications of fractional calculus [37 – 43].

The homotopy analysis method (HAM), initially proposed by Liao in his Ph. D. thesis [44], is a powerful method to solve nonlinear problems [34 – 36, 44 – 49]. The validity of the HAM is independent of whether or not there exist small parameters in the considered equation. The method yields a very rapid convergence of the solution series in most cases, usually only a few iterations lead to very accurate solutions. Here HAM is used to solve linear partial differential equations with fractional order. This method has been successfully applied to solve many types of nonlinear problems by several authors [14, 15, 23 – 25, 44, 49 – 59]. Abbasbandy [51] has investigated an approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by using HAM. Nonlinear fin-type problems have been studied by Chowdhury et al. [51]. This method is used to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity [52]. Domairry et al. [53] have compared HAM and HPM using the nonlinear heat transfer equation. Also, a comparison of HAM and HPM methods was investigated by [54] for solving the nonlinear heat conduction and convection equations. Approximate explicit solutions of nonlinear Benjamin-Bona-Mahony-Burgers (BBMB) equations were found by [55]. In [56] Hayat and Sajid have studied magnetohydrodynamic (MHD) boundary layer flow of an upper-convected Maxwell fluid by using HAM. Linear and nonlinear fractional diffusion-wave equations have been used by [14], also Jafari et al. [15] have investigated a system of nonlinear fractional partial differential equations by using HAM. By using homotopy analysis method, Sajid and Hayat [57] have studied thin film flows of a third-order fluid. Also Song et al. [23] have used the homotopy analysis method for the fractional BBMB equation. Fractional KdV-Burgers-Kuramoto equation has been investigated by [24] where the homotopy analysis method was used. Authors of [58] have studied the nonlinear fractional partial differential equations. Finally, Xu and Jie [25] have investigated analysis of a time fractional wave-like equation and employed the homotopy analysis method. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. More important, the above procedure is just an algebraic algorithm and can be applied in a symbolic computation system so the well-known symbolic software Maple can be used. For some other analytical approaches we refer the interested reader to [61 – 63] for the homotopy perturbation method, to [64, 65] for the variational iteration method, and to [66 – 70] for the Adomian decomposition method.

The current paper is organized as follows: In Section 2, we describe the fractional calculus. In Section 3, the homotopy analysis method will be introduced briefly and this technique will be applied to fractional partial differential equations. Section 4 contains several test problems to show the efficiency and accuracy of the new method, and a conclusion is given in Section 5.

2. Fractional Calculus

Several definitions of fractional calculus have been proposed in the last two centuries. Here, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ on the usual Lebesgue space $L_1[a,b]$ is given by [3]

$$\begin{aligned} \mathrm{J}_a^\alpha f(x) &= \mathrm{D}_a^{-\alpha} f(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad (\alpha > 0), \end{aligned} \tag{1}$$

$$\mathrm{J}_a^0 f(x) = f(x). \tag{2}$$

It has the following properties: (i) J_a^α exists for any $x \in [a,b]$, (ii) $J_a^\alpha \mathrm{J}_a^\beta = \mathrm{J}_a^{\alpha+\beta}$, (iii) $J_a^\alpha \mathrm{J}_a^\beta = \mathrm{J}_a^\beta \mathrm{J}_a^\alpha$, (iv) $J_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$, where $f \in L_1[a,b]$, $\alpha, \beta \geq 0$ and $\gamma > -1$.

It is worth mentioning that the Riemann-Liouville derivative has certain disadvantages for describing some natural phenomena with fractional differential equations. Thus, we introduce Caputo's definition [11] of fractal derivative operator D^α, which is a modification of the Riemann-Liouville definition.

Definition 2. The Caputo definition [11] of fractional derivative operator is given by

$$\begin{aligned} \mathrm{D}_a^\alpha f(x) &= \mathrm{J}_a^{n-\alpha} \mathrm{D}_a^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau, \\ (\alpha > 0), \end{aligned} \tag{3}$$

for $n-1 < \alpha \leq n, n \in \mathbb{N}, x > 0$. It has the following two basic properties for $n-1 < \alpha \leq n$ and $f \in L_1[a,b]$ [11]:

$$\begin{aligned} &\mathrm{D}_a^\alpha \mathrm{J}_a^\alpha f(x) = f(x), \\ &\mathrm{D}\mathrm{J}_a^\alpha \mathrm{D}_a^\alpha f(x) = f(x) - \textstyle\sum_{k=0}^{n-1} f^k(0^+)\frac{(x-a)^k}{k!}, \quad x > 0. \end{aligned} \tag{4}$$

For more mathematical properties of fractional derivatives and integrals, we refer the interested reader to the related references in this subject [2 – 4, 9, 12, 20].

Definition 3. For n being the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as [3]

$$\mathrm{D}_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} \mathrm{d}\tau, \\ \text{if } n-1 < \alpha < n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, \quad \text{if } \alpha = n \in \mathbb{N}. \end{cases} \tag{5}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult [3, 11].

3. Analysis of the Homotopy Analysis Method

In this paper, we apply the homotopy analysis method [49] to solve the linear fractional partial differential equations. This method was proposed by the Chinese mathematician J. S. Liao [44]. We extend Liao's basic ideas to the fractional partial differential equations. Let us consider the fractional partial differential equation

$$\mathcal{FD}(u(x,t)) = 0, \tag{6}$$

where $\mathcal{FD}$ is a fractional partial differential operator, x and t denote independent variables, and $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the same way. Based on the constructed zero-order deformation equation by Liao [49], we give the following zero-order deformation equation in the similar way:

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] = qh\mathcal{FD}(v(x,t;q)), \tag{7}$$

where $q \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear non-integer order operator which possesses the property $\mathcal{L}(C) = 0$, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $v(x,t;q)$ is an unknown function on independent variables x,t,q. It is important to note that one has great freedom to choose the auxiliary parameter h in HAM. If $q = 0$ and $q = 1$, then we have

$$v(x,t;0) = u_0(x,t), \quad v(x,t;1) = u(x,t), \tag{8}$$

respectively. Thus as q increases from 0 to 1, the solution $v(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $v(x,t;q)$ in Taylor series with respect to q, one has

$$v(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{9}$$

where

$$u_m(x,t) = \left. \frac{\partial^m v(x,t;q)}{\partial q^m} \right|_{q=0}. \tag{10}$$

If the auxiliary linear non-integer order operator, the initial guess, and the auxiliary parameter h are so properly chosen, the series (9) converges at $q = 1$. Hence, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{11}$$

which must be one of the solution of the original nonlinear equation as proved by [49]. As $h = -1$, (7) becomes

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] + q\mathcal{FD}v(x,t;q) = 0, \tag{12}$$

which is used mostly in the homotopy perturbation method (HPM). Thus, HPM is a special case of HAM. The comparison between HAM and HPM can be found in [51, 53]. According to (9), the governing equation can be deduced from the zero-order deformation (7). Define the vector

$$\mathbf{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), u_3(x,t), \ldots, u_n(x,t)\}. \tag{13}$$

Differentiating (7) m times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathcal{FR}(\mathbf{u}_{m-1}(x,t)), \tag{14}$$

where

$$\mathcal{FR}(\mathbf{u}_{m-1}(x,t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{FR}(\mathbf{v}(x,t;q))}{\partial q^{m-1}} \right|_{q=0} \tag{15}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{16}$$

Finally, for the purpose of computation, we will approximate the HAM solution (11) by the following truncated series:

$$\phi_m = \sum_{k=0}^{m-1} u_k(x,t). \tag{17}$$

The mth-order deformation Equation (14), is linear and thus can be easily solved.

The convergence of a series is important. A series is often of no use if it is convergent in a rather restricted region. It is clear that the convergence of the series (11) depends upon the auxiliary parameter h, the initial guess $u_0(x,t)$, and the auxiliary linear operator $\mathcal{L}$. Fortunately, the homotopy analysis method provides us with great freedom to choose all of them. A complete review for the convergence discussion is available in [49].

4. Test Problems

In this section, we shall present several test problems to illustrate the applicability of HAM to linear fractional partial differential equations.

Example 1. First, we consider the following non-homogeneous fractional partial differential equation [9]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + c\frac{\partial u}{\partial x} = g(x,t), \quad t>0, \quad 0<\alpha\le 1, \tag{18}$$

where c is a constant and $g(x,t)$, the source term, is a function of x and t. Assume that the initial and boundary conditions are

$$\begin{aligned} &u(x,0) = f(x), \quad x\in\mathbb{R}, \\ &\mathrm{D}u(x,t) \longrightarrow 0 \text{ as } |x|\longrightarrow\infty, \quad t>0. \end{aligned} \tag{19}$$

Now, we use HAM to solve the general non-homogeneous linear equation. To demonstrate the effectiveness of the method, we consider (18) with the initial condition

$$u(x,0) = f(x), \quad c=1. \tag{20}$$

We choose the linear non-integer order operator

$$\mathcal{L}[v(x,t;q)] = \mathrm{D}_t^\alpha v(x,t;q). \tag{21}$$

Furthermore, (18) suggests to define the linear fractional partial differential operator

$$\mathcal{NFD}[v(x,t;q)] = \mathrm{D}_t^\alpha v(x,t;q) + v_x(x,t;q) - g(x,t). \tag{22}$$

Using the above definitions, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] = qh\mathcal{NFD}v(x,t;q). \tag{23}$$

Obviously, when $q=0$ and $q=1$, we can write

$$v(x,t;0) = u_0(x,t) = u(x,0), \quad v(x,t;1) = u(x,t). \tag{24}$$

According to (14)–(16), we gain the mth-order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathcal{FR}(\mathbf{u}_{m-1}(x,t)), \tag{25}$$

where

$$\begin{aligned} \mathcal{FR}(\mathbf{u}_{m-1}(x,t)) = {}& \mathrm{D}_t^\alpha u_{(m-1)} + u_{(m-1)x} \\ &- (1-\chi_m)g(x,t). \end{aligned} \tag{26}$$

Now, the solution of (25), for $m\ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\mathcal{L}^{-1\mathcal{NFR}}(\mathbf{u}_{m-1}(x,t)). \tag{27}$$

From (19), (24), and (27), we now successively obtain

$$u_0(x,t) = u(x,0) = f(x), \tag{28}$$

$$\begin{aligned} u_1(x,t) &= h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^\alpha u_0 + u_{0x} - g(x,t)) \\ &= h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^\alpha f(x) + f_x - g(x,t)) \\ &= h\mathrm{D}_t^{-\alpha}(f_x - g), \end{aligned} \tag{29}$$

$$u_2(x,t) = u_1(x,t) + h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^\alpha u_1 + u_{1x}). \tag{30}$$

With

$$\begin{aligned} &\mathrm{D}_t^\alpha u_1(x,t) = h\mathrm{D}_t^\alpha\mathrm{D}_t^{-\alpha}(f_x - g) \\ &\text{and } \ u_{1x} = h\mathrm{D}_t^{-\alpha}(f_{xx} - g_x), \end{aligned} \tag{31}$$

$$\begin{aligned} u_2(x,t) = {}& h\mathrm{D}_t^{-\alpha}(f_x - g) \\ &+ h\mathrm{D}_t^{-\alpha}(h(f_x-g) + h\mathrm{D}_t^{-\alpha}(f_{xx} - g_x)). \end{aligned} \tag{32}$$

Then we have

$$\begin{aligned} u_2(x,t) = {}& h(h+1)\mathrm{D}_t^{-\alpha}(f_x - g) \\ &+ h^2(\mathrm{D}_t^{-\alpha})^2(f_{xx} - g_x), \end{aligned} \tag{33}$$

$$u_3(x,t) = u_2(x,t) + h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^\alpha u_2 + u_{2x}) \tag{34}$$

$$\begin{aligned} u_3(x,t) = {}& h^3(\mathrm{D}_t^{-\alpha})^3(f_{xxx} - g_{xx}) \\ &+ h^2(h+1)(\mathrm{D}_t^{-\alpha})^2(f_{xx} - g_x) \\ &+ h(h+1)^2\mathrm{D}_t^{-\alpha}(f_x - g) \\ &+ h(h+1)\mathrm{D}_t^{-\alpha}(f_{xx} - g_x), \end{aligned} \tag{35}$$

$$\begin{aligned} u_4(x,t) = {}& h^4(\mathrm{D}_t^{-\alpha})^4(f_{xxxx} - g_{xxx}) \\ &+ 2h^3(h+1)(\mathrm{D}_t^{-\alpha})^3(f_{xxx} - g_{xx}) \\ &+ 2h^2(h+1)^2(\mathrm{D}_t^{-\alpha})^2(f_{xx} - g_x) \\ &+ h^2(h+1)(\mathrm{D}_t^{-\alpha})^2(f_{xxx} - g_{xx}) \\ &+ h(h+1)^2\mathrm{D}_t^{-\alpha}(f_{xx} - g_x) \\ &+ h(h+1)^3\mathrm{D}_t^{-\alpha}(f_x - g), \end{aligned} \tag{36}$$

and so on. If we substitute $h=-1$ in the above terms the dominant terms will be remaining and the rest terms vanish because they include the factor $h^m(h+1)^n,\ m,n\in\mathbb{N}$. Define $A(x,t)=f_x(x)-g(x,t)$, then we have

$$\begin{aligned}
&u_0(x,t)=f(x),\\
&u_1(x,t)=-\mathrm{D}_t^{-\alpha}(A),\\
&u_2(x,t)=(\mathrm{D}_t^{-\alpha})^2(A_x),\\
&u_3(x,t)=-(\mathrm{D}_t^{-\alpha})^3(A_{xx}),\\
&u_4(x,t)=(\mathrm{D}_t^{-\alpha})^4(A_{xxx}),
\end{aligned}\tag{37}$$

and so on. By using (11), we have

$$\begin{aligned}
u(x,t)=f(x)-\mathrm{D}_t^{-\alpha}(A)+(\mathrm{D}_t^{-\alpha})^2(A_x)\\
-(\mathrm{D}_t^{-\alpha})^3(A_{xx})+(\mathrm{D}_t^{-\alpha})^4(A_{xxx})-\ldots
\end{aligned}\tag{38}$$

or

$$u(x,t)=f(x)-\sum_{k=1}^{\infty}(-1)^k(\mathrm{D}_t^{-\alpha})^k(\mathrm{D}_x^{k-1}A).\tag{39}$$

Now, we put $g(x,t)=\exp(-x-t)$ then (18) yields

$$\begin{aligned}
&\mathrm{D}_t^{\alpha}u(x,t)+u_x(x,t)=\exp(-x-t),\\
&u(x,0)=\exp(-x),\\
&u(x,t)\to 0 \ \text{ as } \ |x|\to\infty,\quad t>0.
\end{aligned}\tag{40}$$

Starting with the initial condition $u(x,0)=f(x)=\exp(-x)$, the source term $g(x,t)=\exp(-x-t)$, and the auxiliary operator $\mathcal{L}u(x,t)=\mathrm{D}_t^{\alpha}u(x,t)$. Thus, we have

$$\begin{aligned}
&A=f_x-g=-(\exp(-x)+\exp(-x-t)),\\
&A_x=\exp(-x)+\exp(-x-t),\\
&A_{xx}=A=f_x-g=-(\exp(-x)+\exp(-x-t)),\\
&\mathrm{D}_t^{-\alpha}(A)=\mathrm{D}_t^{-\alpha}\left(-(\exp(-x)+\exp(-x-t))\right)\\
&=-\frac{\exp(-x)}{\Gamma(\alpha+1)}t^{\alpha}+\exp(-x)\mathrm{D}_t^{-\alpha}\exp(-t),\\
&(\mathrm{D}_t^{-\alpha})^2(A_x)=\mathrm{D}_t^{-2\alpha}\left(\exp(-x)+\exp(-x-t)\right)\\
&=\frac{\exp(-x)}{\Gamma(2\alpha+1)}t^{2\alpha}+\exp(-x)\mathrm{D}_t^{-2\alpha}\exp(-t),\\
&(\mathrm{D}_t^{-\alpha})^3(A_{xx})=\mathrm{D}_t^{-3\alpha}\left(-(\exp(-x)+\exp(-x-t))\right)\\
&=-\frac{\exp(-x)}{\Gamma(3\alpha+1)}t^{3\alpha}-\exp(-x)\mathrm{D}_t^{-3\alpha}\exp(-t).
\end{aligned}\tag{41}$$

To solve $\mathrm{D}_t^{-\alpha}\exp(-t)$, the Laplace transform can be used:

$$\mathrm{L}\mathrm{D}_t^{-\alpha}\exp(-t)=\frac{1}{s^{\alpha}(s+1)},\tag{42}$$

and with the use of the inverse Laplace transform we have [9]

$$\begin{aligned}
\mathrm{D}_t^{-\alpha}\exp(-t)&=\mathrm{L}^{-1}\left(\frac{1}{s^{\alpha+1}}\left(1-\frac{1}{s+1}\right)\right)\\
&=\frac{t^{\alpha}}{\Gamma(\alpha+1)}-E(t,\alpha+1,-1),\\
\mathrm{D}_t^{-2\alpha}\exp(-t)&=\mathrm{L}^{-1}\left(\frac{1}{s^{2\alpha+1}}\left(1-\frac{1}{s+1}\right)\right)\\
&=\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-E(t,2\alpha+1,-1),\\
\mathrm{D}_t^{-3\alpha}\exp(-t)&=\mathrm{L}^{-1}\left(\frac{1}{s^{3\alpha+1}}\left(1-\frac{1}{s+1}\right)\right)\\
&=\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-E(t,3\alpha+1,-1),
\end{aligned}\tag{43}$$

where $E(t,\alpha,a)$ is as follows:

$$E(t,\alpha,a)=\frac{1}{\Gamma(\alpha)}\int_0^t\tau^{\alpha-1}\exp(a(t-\tau))\mathrm{d}\tau.\tag{44}$$

With the use of the above formula the solution $u(x,t)$ can be obtained; hence, we have

$$\begin{aligned}
&u(x,t)=\exp(-x)\\
&+2\exp(-x)\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right\}\\
&-\exp(-x)\{E(t,\alpha+1,-1)+E(t,2\alpha+1,-1)\\
&\qquad+E(t,3\alpha+1,-1)+\ldots\}.
\end{aligned}\tag{45}$$

Thus, we get

$$\begin{aligned}
u(x,t)=\exp(-x)+2\exp(-x)\sum_{k=1}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\\
-\exp(-x)\sum_{k=1}^{\infty}E(t,k\alpha+1,-1),
\end{aligned}\tag{46}$$

where for $\alpha=1$ we have

$$\begin{aligned}
E(t,k+1,-1)&=\frac{1}{\Gamma(k+1)}\int_0^t\tau^k\exp(-(t-\tau))\mathrm{d}\tau\\
&=\frac{\exp(-t)}{\Gamma(k+1)}\int_0^t\tau^k\exp(\tau)\mathrm{d}\tau.
\end{aligned}\tag{47}$$

Finally, the exact solution will be

$$u(x,t)=\exp(-x)(\exp(t)+\sinh(t)).\tag{48}$$

Example 2. Consider the following linear non-homogeneous fractional Burgers equation [9]:

$$\mathrm{D}_t^{\alpha}u(x,t)+c\frac{\partial u}{\partial x}(x,t)-b\frac{\partial^2 u}{\partial x^2}(x,t)=g(x,t), \quad x\in\mathbb{R}, \quad t>0, \tag{49}$$

where c is a constant, $0<\alpha\le 1$, b is cinematic and $g(x,t)$, the source term, is a function of x and t. We use the initial and boundary conditions as

$$\begin{aligned} &u(x,0)=f(x), \quad x\in\mathbb{R},\\ &u(x,t)\longrightarrow 0 \ \text{as} \ |x|\longrightarrow\infty, \quad t>0. \end{aligned} \tag{50}$$

By defining the linear non-integer order operator we get

$$\mathcal{L}[v(x,t;q)]=\mathrm{D}_t^{\alpha}v(x,t;q). \tag{51}$$

Furthermore, (49) suggests to define the linear fractional partial differential operator

$$\begin{aligned}\mathcal{NFD}[v(x,t;q)]=\mathrm{D}_t^{\alpha}v(x,t;q)+cv_x(x,t;q)\\ -bv_{xx}(x,t;q)-g(x,t).\end{aligned} \tag{52}$$

By manipulating the procedure presented in Example 1, we gain the mth-order fractional equation

$$\begin{aligned}\mathcal{NFR}(\mathbf{u}_{m-1}(x,t))=\mathrm{D}_t^{\alpha}u_{(m-1)}+cu_{(m-1)x}\\ -bu_{(m-1)xx}-(1-\chi_m)g(x,t).\end{aligned} \tag{53}$$

Now, for $m\ge 1$ we have

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+h\mathcal{L}^{-1\mathcal{NFR}}(\mathbf{u}_{m-1}(x,t)). \tag{54}$$

From (53) and (54), we now successively obtain

$$u_0(x,t)=u(x,0)=f(x), \tag{55}$$

$$u_1(x,t)=h\mathrm{D}_t^{-\alpha}(cf_x(x)-bf_{xx}(x)-g(x,t)). \tag{56}$$

Define $A(x,t)=cf_x(x)-bf_{xx}(x)-g(x,t)$, then u_2, u_3, $u_4,\ldots$ will be obtained as follows:

$$u_2(x,t)=h^2(\mathrm{D}_t^{-\alpha})^2(cA_x-bA_{xx})+h(h+1)\mathrm{D}_t^{-\alpha}(A),$$

$$\begin{aligned}u_3(x,t)=h^3(\mathrm{D}_t^{-\alpha})^3(c^2A_{xx}-2cbA_{xxx}+b^2A_{xxxx})\\ +2h^2(h+1)(\mathrm{D}_t^{-\alpha})^2(cA_x-bA_{xx})\\ +h(h+1)^2\mathrm{D}_t^{-\alpha}(A),\end{aligned}$$

$$\begin{aligned}&u_4(x,t)=u_3+h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^{\alpha}u_3+cu_{3x}-bu_{3xx})=\\ &h^4(\mathrm{D}_t^{-\alpha})^4(c^3A_{xxx}-3c^2bA_{xxxx}\\ &\qquad+3cb^2A_{xxxxx}-b^3A_{xxxxxx})\\ &+3h^3(h+1)(\mathrm{D}_t^{-\alpha})^3(c^2A_{xx}-2cbA_{xxx}+b^2A_{xxxx})\\ &+3h^2(h+1)^2(\mathrm{D}_t^{-\alpha})^2(cA_x-bA_{xx})\\ &+h(h+1)^3\mathrm{D}_t^{-\alpha}(A),\end{aligned} \tag{57}$$

and so on. Define

$$\begin{aligned}&K_0=A(x,t), \quad K_1=cA_x-bA_{xx},\\ &K_2=c^2A_{xx}-2bcA_{xxx}+b^2A_{xxxx},\end{aligned} \tag{58}$$

$$K_m(x,t)=\sum_{r=0}^{m}(-1)^r b^r c^{m-r}\binom{m}{r}\frac{\mathrm{d}^{m+r}}{\mathrm{d}x^{m+r}}K_0(x,t), \tag{59}$$

$$T_m(x,t)=\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}K_{m-1}(x,t), \tag{60}$$

thus, the exact solution is as follows:

$$u_m(x,t)=\sum_{k=0}^{m-1}\binom{m-1}{k}h^{k+1}(h+1)^{m-1-k}T_m(x,t), \quad m\ne 0, \tag{61}$$

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{62}$$

Consider $h=-1$ and using (11), we have

$$\begin{aligned}u(x,t)=f(x)-\mathrm{D}_t^{-\alpha}(A)+(\mathrm{D}_t^{-\alpha})^2(cA_x-bA_{xx})\\ -(\mathrm{D}_t^{-\alpha})^3(c^2A_{xx}-2cbA_{xxx}+b^2A_{xxxx})\\ +(\mathrm{D}_t^{-\alpha})^4(c^3A_{xxx}-2c^2bA_{xxxx}\\ +3cb^2A_{xxxxx}-b^3A_{xxxxxx})-\ldots,\end{aligned} \tag{63}$$

where the solution of the problem is as follows:

$$u(x,t)=\sum_{n=0}^{\infty}(-1)^n(\mathrm{D}_t^{-\alpha})^n K_n(x,t). \tag{64}$$

Now, if $g(x,t)=0$ and $c=b=1$ in (49), then we have:

$$\begin{aligned}&\mathrm{D}_t^{\alpha}u(x,t)+u_x(x,t)-u_{xx}(x,t)=0,\\ &u(x,0)=\exp(-x),\\ &u(x,t)\to 0 \ \text{as} \ |x|\to\infty, \quad t>0.\end{aligned} \tag{65}$$

Starting with the initial condition $u(x,0)=f(x)=\exp(-x)$, the source term $g(x,t)=0$, and the auxiliary operator $\mathcal{L}u(x,t)=\mathrm{D}_t^{\alpha}u(x,t)$, we have

$$\begin{aligned}&f(x)=\exp(-x), \quad g(x,t)=0,\\ &A=-2\exp(-x), \quad A_x=2\exp(-x), \ \ldots,\\ &A_x^{(n)}=-2(-1)^n\exp(-x).\end{aligned} \tag{66}$$

Note that $A_x^{(n)}$ indicates the derivative of order n with respect to x. Thus, we obtain

$$u_0=\exp(-x),$$

$$u_1=-\mathrm{D}_t^{-\alpha}(-2\exp(-x))=\frac{2}{\Gamma(\alpha+1)}\exp(-x)t^{\alpha},$$

$$u_2=(\mathrm{D}_t^{-\alpha})^2(4\exp(-x))=\frac{4}{\Gamma(2\alpha+1)}\exp(-x)t^{2\alpha},$$

(a) (b)

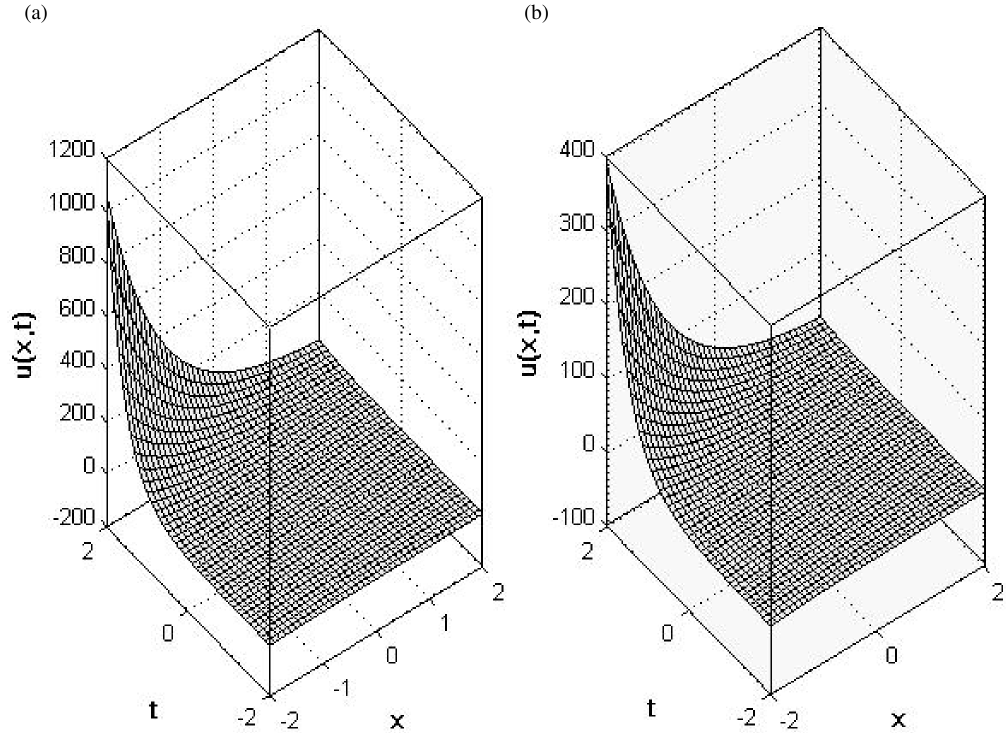

Fig. 1. 22nd-order approximation solution of u to (65) when $h = -1$ (a) $\alpha = 0.8$, (b) $\alpha = 1$.

(c) (d)

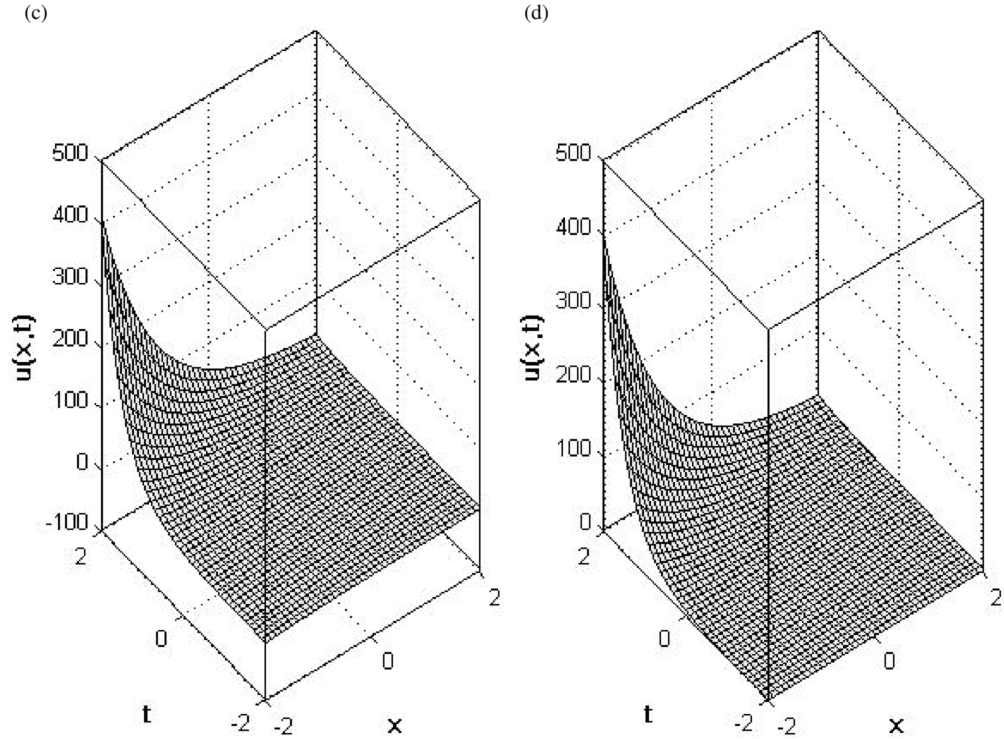

Fig. 2. 22nd-order approximation and exact solution of u to (65) when $h = -1$ (c) $\alpha = 0.99$, (d) exact ($\alpha = 1$).

Table 1. Approximate solution of (65) for some values of h using the 11-term HAM approximation ϕ_{11} with $\alpha = 1$.

(x,t)	$h=-0.62$	$h=-0.75$	$h=-1$	$h=-1.5$	$h=-1.75$	exact
(0.1,0.1)	1.105126652	1.105169444	1.105170918	1.105202558	1.106418985	1.105170918
(0.1,0.2)	1.349622203	1.349846929	1.349858808	1.349852637	1.352130933	1.349858808
(0.1,0.3)	1.647926973	1.648667790	1.648721270	1.648639165	1.645898502	1.648721271
(0.1,0.4)	2.011627299	2.013572229	2.013752705	2.013736080	2.007466736	2.013752707
(0.1,0.5)	2.454660538	2.459094031	2.459603085	2.459751366	2.456874977	2.459603111

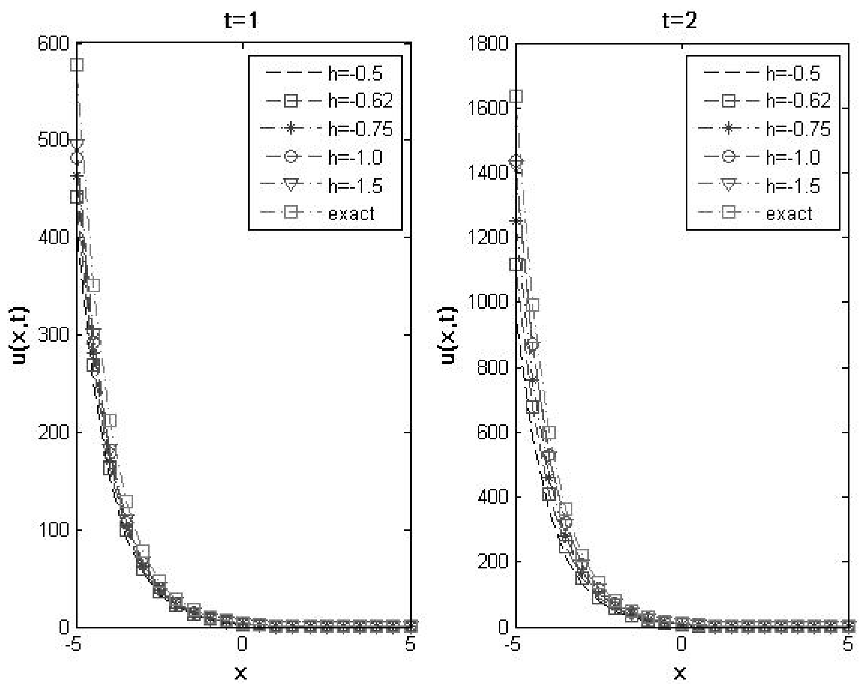

Fig. 3. 4th-order approximation solution of u to (82) with $c = 1$, $f(x) = \exp(-x)$, $g(x,t) = \exp(-x-t)$, and $\alpha = 1$.

$$u_3 = -(\mathrm{D}_t^{-\alpha})^3(-8\exp(-x)) = \frac{8}{\Gamma(3\alpha+1)}\exp(-x)t^{3\alpha}, \tag{67}$$

and so on. Therefore, we get

$$\begin{aligned} u(x,t) &= \exp(-x) + \frac{2}{\Gamma(\alpha+1)}\exp(-x)t^{\alpha} \\ &\quad + \frac{4}{\Gamma(2\alpha+1)}\exp(-x)t^{2\alpha} \\ &\quad + \frac{8}{\Gamma(3\alpha+1)}\exp(-x)t^{3\alpha} + \ldots \\ &= \exp(-x)\sum_{n=0}^{\infty}\frac{2^n t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{68}$$

where for $\alpha = 1$, we have

$$u(x,t) = \exp(2t - x). \tag{69}$$

Example 3. Consider the linear non-homogeneous fractional KdV equation [9] as follows:

$$\mathrm{D}_t^{\alpha}u(x,t) + c\frac{\partial u}{\partial x}(x,t) + b\frac{\partial^3 u}{\partial x^3}(x,t) = g(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{70}$$

where b and c are constants, $0 < \alpha \le 1$, and $g(x,t)$, the source term, is a function of x and t. We assume that the initial and boundary conditions are

$$\begin{aligned} &u(x,0) = f(x), \quad x \in \mathbb{R}, \\ &u(x,t) \to 0 \ \text{as} \ |x| \to \infty, \quad t > 0. \end{aligned} \tag{71}$$

By manipulating the procedure introduced in Example 1, we define the linear operators as

$$\mathcal{L}[v(x,t;q)] = \mathrm{D}_t^{\alpha}v(x,t;q), \tag{72}$$

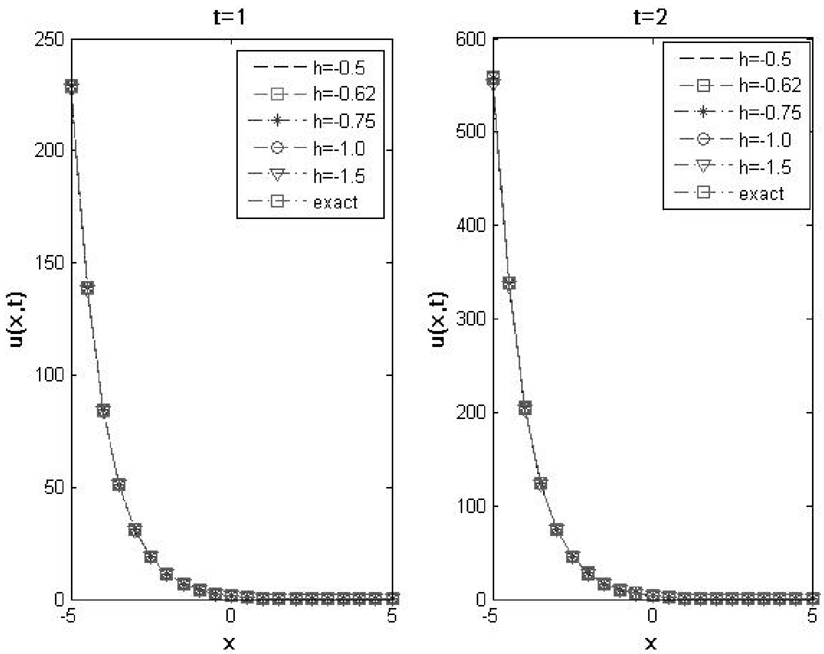

Fig. 4. 5th-order approximation solution of u to (83) with $\alpha = 1$.

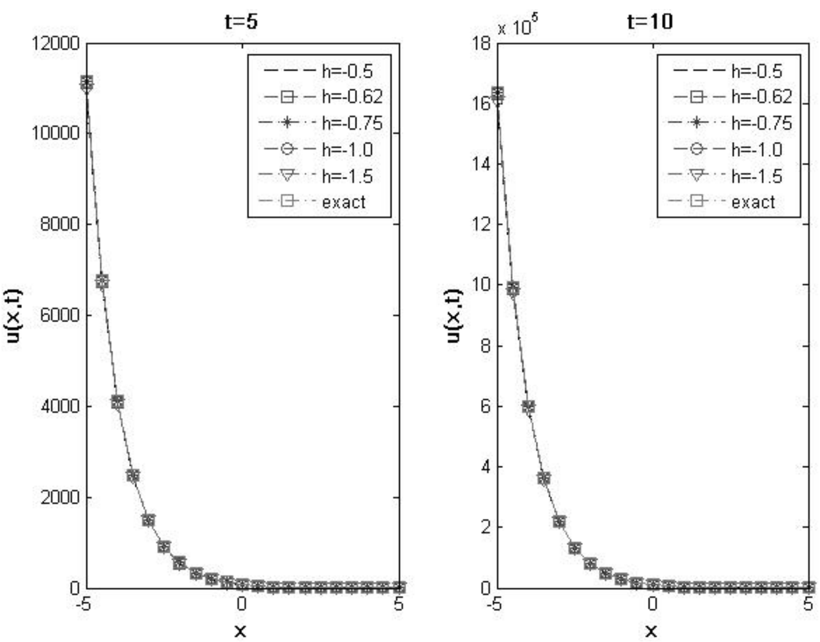

Fig. 5. 5th-order approximation solution of u to (86) with $\alpha = 1$.

$$\mathcal{NFD}[v(x,t;q)] = \mathrm{D}_t^{\alpha} v(x,t;q) + c v_x(x,t;q) + b v_{xxx}(x,t;q) - g(x,t). \tag{73}$$

Using the above definition, we gain the mth-order linear fractional operator as follows:

$$\mathcal{NFR}(\mathbf{u}_{m-1}(x,t)) = \mathrm{D}_t^{\alpha} u_{(m-1)} + c u_{(m-1)x} + b u_{(m-1)xxx} - (1-\chi_m) g(x,t). \tag{74}$$

Consequently, the first few terms of the HAM series solution are given in the following:

$$u_0(x,t) = u(x,0) = f(x), \tag{75}$$

Table 2. Approximate solution of (83) for some values of h using the 11-term HAM approximation ϕ_{11} with $\alpha = 1$.

(x,t)	$h=-0.5$	$h=-0.62$	$h=-0.75$	$h=-1$	$h=-1.5$	exact
(0.1,0.1)	0.9093631655	0.9093652684	0.9093653754	0.9093653765	0.9093675874	0.9093653765
(0.1,0.2)	0.9229857039	0.9229941365	0.9229945654	0.9229945697	0.9230034355	0.9229945697
(0.1,0.3)	0.9458413707	0.9458604232	0.9458613922	0.9458614019	0.9458814332	0.9458614019
(0.1,0.4)	0.9781589148	0.9781929836	0.9781947162	0.9781947338	0.9782305528	0.9781947338
(0.1,0.5)	1.020261780	1.020315412	1.020318139	1.020318167	1.020374554	1.020318167

Table 3. Approximate solution of (86) for some values of h using the 11-term HAM approximation ϕ_{11} with $\alpha = 1$.

(x,t)	$h=-0.5$	$h=-0.62$	$h=-0.75$	$h=-1$	$h=-1.5$	exact
(0.1,0.1)	0.9954277863	0.9954698791	0.9954720198	0.9954720414	0.9955162966	0.9954720414
(0.1,0.2)	1.086924813	1.087009420	1.087013723	1.087013767	1.087102720	1.087013766
(0.1,0.3)	0.9089255515	1.180372200	1.180378708	1.180378774	1.180513316	1.180378774
(0.1,0.4)	1.276320015	1.276492625	1.276501403	1.276501492	1.276682968	1.276501492
(0.1,0.5)	1.376113721	1.376332700	1.376343836	1.376343949	1.376574176	1.376343949

Table 4. Absolute error $|u-\phi_{11}|$ for the (65) with $h=-1$ and $\alpha = 1$.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.721×10^{-16}	9.834×10^{-13}	8.655×10^{-11}	2.085×10^{-9}	2.471×10^{-8}
0.2	4.271×10^{-16}	8.898×10^{-13}	7.831×10^{-11}	1.887×10^{-9}	2.236×10^{-8}
0.3	3.865×10^{-16}	8.052×10^{-13}	7.076×10^{-11}	1.707×10^{-9}	2.023×10^{-8}
0.4	3.497×10^{-16}	7.285×10^{-13}	6.412×10^{-11}	1.545×10^{-9}	1.831×10^{-8}
0.5	3.165×10^{-16}	6.592×10^{-13}	5.802×10^{-11}	1.398×10^{-9}	1.657×10^{-8}

$$u_1(x,t) = h\mathrm{D}_t^{-\alpha}(cf_x(x) + bf_{xxx}(x) - g(x,t)). \tag{76}$$

Define $A(x,t) = cf_x(x) + bf_{xxx}(x) - g(x,t)$,

$$\begin{aligned} &K_0 = f(x), \quad K_1 = A, \quad K_2 = cA_x + bA_{xxx}, \\ &K_i = c\frac{\partial K_{i-1}}{\partial x} + b\frac{\partial K_{i-1}}{\partial x^3} \text{ for } i = 3,4,5,\ldots, \end{aligned} \tag{77}$$

then u_2, u_3, u_4, … will be obtained in the following form:

$$\begin{aligned} u_0 &= K_0, \\ u_1 &= h\mathrm{D}_t^{-\alpha}(K_1), \\ u_2 &= h^2(\mathrm{D}_t^{-\alpha})^2(K_2) + h(h+1)\mathrm{D}_t^{-\alpha}(K_1), \\ u_3 &= h^3(\mathrm{D}_t^{-\alpha})^3(K_3) + 2h^2(h+1)(\mathrm{D}_t^{-\alpha})^2(K_2) \\ &\quad + h(h+1)^2\mathrm{D}_t^{-\alpha}(K_1), \end{aligned} \tag{78}$$

$$\begin{aligned} u_4 &= u_3 + h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^{\alpha}u_3 + cu_{3x} + bu_{3xxx}) \\ &= (h+1)u_3 + h\mathrm{D}_t^{-\alpha}(cu_{3x} + bu_{3xxx}), \end{aligned} \tag{79}$$

and we note that

$$\begin{aligned} h\mathrm{D}_t^{-\alpha}u_{3x} &= h^4(\mathrm{D}_t^{-\alpha})^4(K_{3x}) + 2h^3(h+1)(\mathrm{D}_t^{-\alpha})^3(K_{2x}) \\ &\quad + h^2(h+1)^2(\mathrm{D}_t^{-\alpha})^2(K_{1x}), \\ u_4 &= h^4(\mathrm{D}_t^{-\alpha})^4(K_4) + 3h^3(h+1)(\mathrm{D}_t^{-\alpha})^3(K_3) \\ &\quad + 3h^2(h+1)^2(\mathrm{D}_t^{-\alpha})^2(K_2) + h(h+1)^3\mathrm{D}_t^{-\alpha}(K_1), \end{aligned} \tag{80}$$

and so on. Thus the solution of the KdV equation is as follows:

$$\begin{aligned} u(x,t) &= K_0 + h\mathrm{D}_t^{-\alpha}(K_1) + h^2(\mathrm{D}_t^{-\alpha})^2(K_2) \\ &\quad + h(h+1)\mathrm{D}_t^{-\alpha}(K_1) + h^3(\mathrm{D}_t^{-\alpha})^3(K_3) \\ &\quad + 2h^2(h+1)(\mathrm{D}_t^{-\alpha})^2(K_2) \\ &\quad + h(h+1)^2\mathrm{D}_t^{-\alpha}(K_1). \end{aligned} \tag{81}$$

Consider $h=-1$, then we obtain

$$\begin{aligned} u(x,t) &= K_0 - \mathrm{D}_t^{-\alpha}(K_1) + (\mathrm{D}_t^{-\alpha})^2(K_2) \\ &\quad - (\mathrm{D}_t^{-\alpha})^3(K_3) + \ldots \\ &= \sum_{n=0}^{\infty}(-1)^n(\mathrm{D}_t^{-\alpha})^n K_n. \end{aligned} \tag{82}$$

Now, if $g(x,t) = \exp(-x)\sinh(t)$, $c = 1$, and $b = -1$ in (70), then we have

$$\begin{aligned} &\mathrm{D}_t^{\alpha}u(x,t) + u_x(x,t) - u_{xxx}(x,t) = \exp(-x)\sinh(t), \\ &u(x,0) = \exp(-x), \\ &u(x,t) \to 0 \text{ as } |x| \to \infty, \quad t > 0. \end{aligned} \tag{83}$$

Starting with the initial condition $u(x,0) = f(x) = \exp(-x)$, the source term $g(x,t) = \exp(-x)\sinh(t)$, and the auxiliary operator $\mathcal{L}u(x,t) = \mathrm{D}_t^{\alpha}u(x,t)$, and using (77), we have

$$K_0 = f(x) = \exp(-x),$$
$$K_1 = f_x - f_{xxx} - \exp(-x)\sinh(t) = -\exp(-x)\sinh(t),$$
$$K_2 = K_3 = \ldots = 0. \quad (84)$$

Thus, $u(x,t)$ is as follows:

$$\begin{aligned} u(x,t) &= \exp(-x) + \exp(-x)\mathrm{D}_t^{-\alpha}\sinh(t) \\ &= \exp(-x) + \frac{1}{2}\exp(-x)\{E(t,\alpha+1,1) \\ &\quad + E(t,\alpha+1,-1)\} \\ &= \exp(-x)\left\{1 + \frac{1}{\Gamma(\alpha+1)}\int_0^t \tau^\alpha \cosh(t-\tau)\mathrm{d}\tau\right\}. \end{aligned} \quad (85)$$

From (85) and $\alpha = 1$, hence, we obtain

$$u(x,t) = \exp(-x)\cosh(t). \quad (86)$$

Also, if we put $g(x,t) = \exp(-x)\cosh(t)$, $c = 1$, and $b = -1$ in (70), then we have:

$$\begin{aligned} &f(x) = \exp(-x), \quad g(x,t) = \exp(-x)\cosh(t), \\ &K_0 = f(x) = \exp(-x), \\ &K_1 = A = f_x - f_{xxx} - g = -\exp(-x)\cosh(t), \\ &K_2 = K_3 = \ldots = 0. \end{aligned} \quad (87)$$

Note that

$$\begin{aligned} &\mathrm{D}_t^{-\alpha}\cosh(t) = \\ &\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_0^t \tau^\alpha \sinh(t-\tau)\mathrm{d}\tau, \\ &u_0 = \exp(-x), \\ &u_1 = \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_0^t \tau^\alpha \sinh(t-\tau)\mathrm{d}\tau\right)\exp(-x), \ldots \end{aligned} \quad (88)$$

Thus, $u(x,t)$ is as follows:

$$\begin{aligned} u(x,t) = \exp(-x) + \exp(-x)\Bigg\{&\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{1}{\Gamma(\alpha+1)}\int_0^t \tau^\alpha \sinh(t-\tau)\mathrm{d}\tau\Bigg\}, \end{aligned} \quad (89)$$

where for $\alpha = 1$ the following solution will be obtained:

$$u(x,t) = \exp(-x)(1+\sinh(t)). \quad (90)$$

Example 4. Consider the linear fractional nonhomogeneous KdV-Burgers equation [9] which is given in the following:

$$\begin{aligned} &\mathrm{D}_t^\alpha u(x,t) + c\frac{\partial u}{\partial x}(x,t) - d\frac{\partial^2 u}{\partial x^2}(x,t) + b\frac{\partial^3 u}{\partial x^3}(x,t) \\ &= g(x,t), \quad x \in \mathbb{R},\ t > 0, \end{aligned} \quad (91)$$

where b, c, and d are the constants, $0 < \alpha \le 1$, and $g(x,t)$, the source term, is a function of x and t. Assume that the initial and boundary conditions are

$$\begin{aligned} &u(x,0) = f(x), \quad x \in \mathbb{R}, \\ &u(x,t) \to 0 \text{ as } |x| \to \infty, \quad t > 0. \end{aligned} \quad (92)$$

By manipulating the procedure mentioned in Example 1, we define the linear operators as:

$$\mathcal{L}[v(x,t;q)] = \mathrm{D}_t^\alpha v(x,t;q). \quad (93)$$

$$\begin{aligned} &\mathcal{NFD}[v(x,t;q)] = \mathrm{D}_t^\alpha v(x,t;q) + cv_x(x,t;q) \\ &-dv_{xx}(x,t;q) + bv_{xxx}(x,t;q) - g(x,t). \end{aligned} \quad (94)$$

Using the above definition, we gain the mth-order linear fractional operator as follows:

$$\begin{aligned} &\mathcal{NFR}(\mathbf{u}_{m-1}(x,t)) = \mathrm{D}_t^\alpha u_{(m-1)} + cu_{(m-1)x} \\ &-du_{(m-1)xx} + bu_{(m-1)xxx} - (1-\chi_m)g(x,t). \end{aligned} \quad (95)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = u(x,0) = f(x), \quad (96)$$

$$\begin{aligned} u_1(x,t) = h\mathrm{D}_t^{-\alpha}\big[&\mathrm{D}_t^\alpha f(x) + cf_x(x) - df_{xx}(x) \\ &+ bf_{xxx}(x) - g(x,t)\big]. \end{aligned} \quad (97)$$

Define

$$\begin{aligned} &K_0(x,t) = f(x), \\ &K_1(x,t) = cf_x(x) - df_{xx}(x) + bf_{xxx}(x) - g(x,t), \\ &K_i(x,t) = cK_{(i-1)x}(x,t) - dK_{(i-1)xx}(x,t) \\ &\qquad + bK_{(i-1)xxx}(x,t), \end{aligned} \quad (98)$$

then

$$\begin{aligned} &u_2(x,t) = h^2(\mathrm{D}_t^{-\alpha})^2 K_2 + h(h+1)\mathrm{D}_t^{-\alpha}K_1, \\ &u_3(x,t) = h^3(\mathrm{D}_t^{-\alpha})^3 K_3 + 2h^2(h+1)(\mathrm{D}_t^{-\alpha})^2 K_2 \\ &\qquad + h(h+1)^2\mathrm{D}_t^{-\alpha}K_1, \end{aligned} \quad (99)$$

and so on. Thus, for $h = -1$, the solution is as follows:

$$u(x,t) = \sum_{n=0}^{\infty}(-1)^n(\mathrm{D}_t^{-\alpha})^n K_n. \quad (100)$$

Table 5. Absolute error $|u-\phi_{16}|$ for (101) with $h=-1$ and $\alpha=1$.

t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	2.527×10^{-22}	1.686×10^{-17}	1.128×10^{-14}	1.147×10^{-12}	4.152×10^{-11}
0.2	2.286×10^{-22}	1.526×10^{-17}	1.021×10^{-14}	1.038×10^{-12}	3.757×10^{-11}
0.3	2.069×10^{-22}	1.381×10^{-17}	9.235×10^{-15}	9.388×10^{-13}	3.399×10^{-11}
0.4	1.872×10^{-22}	1.249×10^{-17}	8.357×10^{-15}	8.495×10^{-13}	3.056×10^{-11}
0.5	1.694×10^{-22}	1.131×10^{-17}	7.561×10^{-15}	7.686×10^{-13}	2.783×10^{-11}

Table 6. Approximate solution of (117) for some values of h using the 11-term HAM approximation ϕ_{11} with $\alpha=2$.

(x,t)	$h=-0.5$	$h=-0.62$	$h=-0.75$	$h=-1$	$h=-1.5$	exact
(0.1,0.1)	0.9049882260	0.9049882960	0.9049882996	0.9049882996	0.9049883733	0.9049882996
(0.1,0.2)	0.9060456928	0.9060462542	0.9060462828	0.9060462831	0.9060468733	0.9060462826
(0.1,0.3)	0.9089255515	0.9089274510	0.9089275476	0.9089275486	0.9089295457	0.9089275482
(0.1,0.4)	0.9145617742	0.9145662926	0.9145665224	0.9145665247	0.9145712753	0.9145665249
(0.1,0.5)	0.9239159195	0.9239247844	0.9239252352	0.9239252398	0.9239345600	0.9239252402

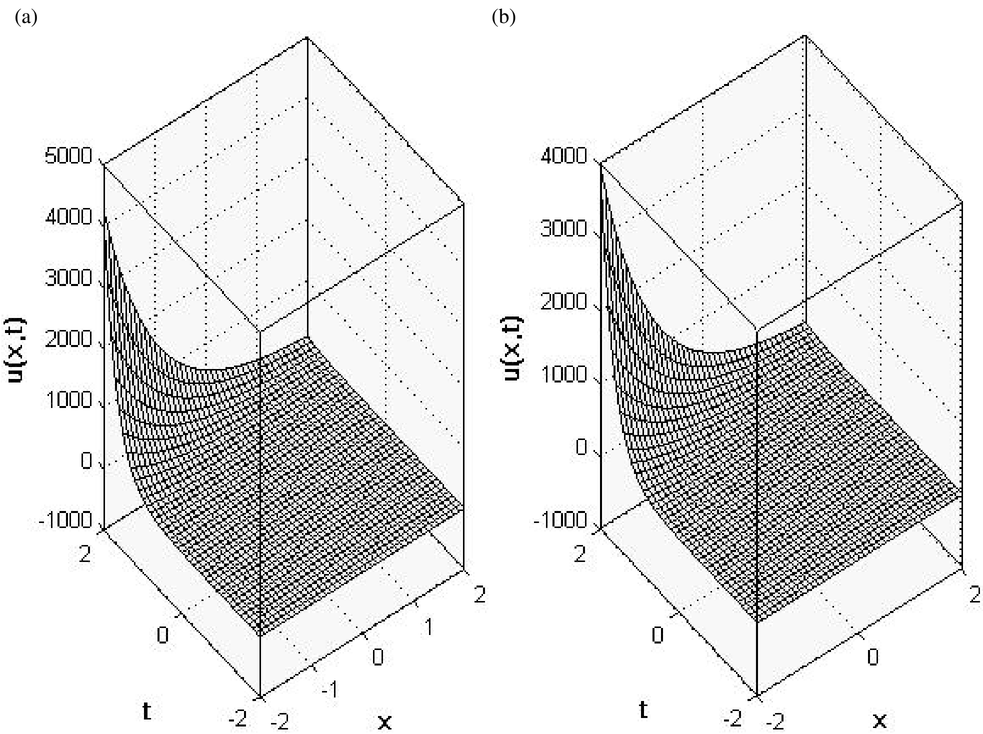

Fig. 6. 14th-order approximation solution of u to (101) with $h=-1$ (a) $\alpha=0.99$, (b) exact ($\alpha=1$).

Now, consider the following example:

$$\begin{aligned}&\mathrm{D}_t^{\alpha}u(x,t)+\frac{\partial u}{\partial x}(x,t)-\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\partial^3 u}{\partial x^3}(x,t)\\&=\exp(-x),\qquad x\in\mathbb{R},\ t>0,\\&u(x,0)=\exp(-x),\\&u(x,t)\to 0\ \text{ as }\ |x|\to\infty,\quad t>0.\end{aligned}\tag{101}$$

We use the initial condition $u(x,0)=f(x)=\exp(-x)$, the source term $g(x,t)=\exp(-x)$, and the auxiliary operator $\mathcal{L}u(x,t)=\mathrm{D}_t^{\alpha}u(x,t)$. Using (98), we have

$$\begin{aligned}&K_0=f(x)=\exp(-x),\\&K_1=f_x-f_{xx}+f_{xxx}-g=-4\exp(-x),\\&K_2=12\exp(-x),K_3=-36\exp(-x),\ldots.\end{aligned}\tag{102}$$

Applying the above procedure yields

$$u(x,t)=\frac{1}{3}\exp(-x)\left\{4\sum_{n=0}^{\infty}\frac{3^n t^{n\alpha}}{\Gamma(n\alpha+1)}-1\right\},\tag{103}$$

where for $\alpha = 1$ it is

$$u(x,t) = \frac{1}{3}\exp(-x)(4\exp(3t) - 1), \tag{104}$$

which is the exact solution.

Example 5. As the last example, we consider the following linear non-homogeneous fractional Klein-Gordon equation [9]:

$$\begin{aligned} &\mathrm{D}_t^{\alpha} u(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) + d^2 u(x,t) = g(x,t), \\ &x \in \mathbb{R}, \quad t > 0, \end{aligned} \tag{105}$$

where c and d are constant, $1 < \alpha \leq 2$, and $g(x,t)$, the source term, is a function of x and t. We assume that the initial and boundary conditions are as follows:

$$\begin{aligned} &u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x), \quad x \in \mathbb{R}, \\ &u(x,t) \to 0 \text{ as } |x| \to \infty, \quad t > 0. \end{aligned} \tag{106}$$

By manipulating the above procedure we define the linear operators as follows:

$$\mathcal{L}[v(x,t;q)] = \mathrm{D}_t^{\alpha} v(x,t;q). \tag{107}$$

$$\begin{aligned} \mathcal{NFD}[v(x,t;q)] = {} & \mathrm{D}_t^{\alpha} v(x,t;q) - c^2 v_{xx}(x,t;q) \\ & + d^2 v(x,t;q) - g(x,t). \end{aligned} \tag{108}$$

Using the above definition, we gain the mth-order linear fractional operator as follows:

$$\begin{aligned} \mathcal{NFR}(\mathbf{u}_{m-1}(x,t)) = {} & \mathrm{D}_t^{\alpha} u_{(m-1)} - c^2 u_{(m-1)xx} \\ & + d^2 u_{(m-1)} - (1-\chi_m) g(x,t). \end{aligned} \tag{109}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = u(x,0) = f(x), \tag{110}$$

$$\begin{aligned} u_1(x,t) = {} & h\mathrm{D}_t^{-\alpha}(\mathrm{D}_t^{\alpha} f(x) - c^2 f_{xx}(x) \\ & + d^2 f(x) - g(x,t)). \end{aligned} \tag{111}$$

Define

$$\begin{aligned} &K_0(x,t) = f(x), \\ &K_1(x,t) = -c^2 f_{xx}(x) + d^2 f(x) - g(x,t), \\ &K_i(x,t) = -c^2 K_{(i-1)xx}(x,t) + d^2 K_{i-1}(x,t), \end{aligned} \tag{112}$$

then

$$\begin{aligned} u_2(x,t) = {} & h^2(\mathrm{D}_t^{-\alpha})^2 K_2 + h(h+1)\mathrm{D}_t^{-\alpha} K_1, \\ u_3(x,t) = {} & h^3(\mathrm{D}_t^{-\alpha})^3 K_3 + 2h^2(h+1)(\mathrm{D}_t^{-\alpha})^2 K_2 \\ & + h(h+1)^2 \mathrm{D}_t^{-\alpha} K_1, \end{aligned} \tag{113}$$

and so on. Thus, for $h = -1$, the solution is as follows:

$$u(x,t) = \sum_{n=0}^{\infty} (-1)^n (\mathrm{D}_t^{-\alpha})^n K_n. \tag{114}$$

We use the initial condition $u(x,0) = f(x) = \exp(-x)$, the source term $g(x,t) = \exp(-x)\sinh(t)$, and the auxiliary operator $\mathcal{L}u(x,t) = \mathrm{D}_t^{\alpha} u(x,t)$. Notice that in (105), $c = d = 1$, $h = -1$. Thus, we have

$$\begin{aligned} &K_0 = f(x) = \exp(-x), \\ &K_1 = -f_{xx} + f - g = -\exp(-x)\sinh(t), \\ &K_2 = K_3 = K_4 = \ldots = 0. \end{aligned} \tag{115}$$

Applying the above procedure, yields

$$\begin{aligned} &u(x,t) = \\ &\exp(-x)\left\{1 + \frac{1}{\Gamma(\alpha+1)} \int_0^t \tau^{\alpha} \cosh(t-\tau) \mathrm{d}\tau \right\}, \end{aligned} \tag{116}$$

where for $\alpha = 2$ we habe

$$u(x,t) = \exp(-x)\left(\sinh(t) - t + 1\right), \tag{117}$$

which is the exact solution. The parameter h determines the convergence region and rate of the approximation for HAM which is shown in Tables 1 – 3 and 6. If we take $h = -1$, we obtain the exact results which are presented in these tables. Tables 1 – 3 and 6 show the 11-term HAM approximate solutions ϕ_{11} of (65), (83), (86), and (117) for different values of h. Tables 4 and 5 show the approximate errors of (65) and (101), respectively, with $h = -1$ and $\alpha = 1$. It is clear that when we take $h = -1$, we obtain the best results for the case $\alpha = 1$ which has an exact solution. In Figures 1, 2, and 6 we plot the approximate solutions for various α and $h = -1$. In Figures 3, 4, and 5 we plot the approximate solutions for various h and $\alpha = 1$.

5. Conclusion

In this paper, fractional wave, Burgers, KdV, KdV-Burgers, and Klein-Gordon equations, were investigated and by using the homotopy analysis method the exact solutions were obtained. The fractional derivative operator in (7) is a linear operator. Based on the homotopy analysis method (HAM), a new analytic technique is proposed to solve the linear fractional partial differential equations. It provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter h.

This is an obvious advantage of the homotopy analysis method. This work illustrates the validity and great potential of the homotopy analysis method for linear fractional partial differential equations. In this way, we obtained solutions in power series. However, it is well known that a power series often has a small convergence radius. It should be emphasized that, in the frame of the homotopy analysis method, we have great freedom to choose the initial guess and the auxiliary linear operator. This work shows that the homotopy analysis method is a very efficient and powerful tool for solving the linear fractional partial differential equations of various types.

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